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Averaged lagrangians containing higher derivatives

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Abstract. The theory of averaged lagrangians is extended to the case of lagrangians containing higher derivatives. The frequency equation and averaged stored energy of oscillatory systems are expressed in terms of the averaged lagrangian. The relation between averaged energy and the averaged hamiltonian is discussed and finally the analysis is applied to two well known mechanical examples and compared with the conventional approach.

1. Introduction

The success of an approach to linear and nonlinear wave problems based on an averaged lagrangian density as the fundamental quantity is well known. Whitham (1967) first introduced the approach in connection with the nonlinear interaction of water waves. Subsequently the method has been used in many applications in a great variety of fields; for example, to the interaction between hydromagnetic waves (Dewar 1970), to plasma wave interactions (Boyd and Turner 1972), to various water wave problems (eg Simmons 1969), and to the astrophysical theory of density waves in the spiral arms of the galaxies (Dewar 1972). The method has also been used in general investigations of wave problems (eg Bretherton and Garrett 1969, Hayes 1970, Askne 1972a).

However, these theories have been confined to the case of lagrangians which contain first derivatives only of the field variable. On the other hand, there has recently been a marked interest in lagrangians of a more general form. Several attempts have been made to formulate a consistent field theory in terms of lagrangians containing higher derivatives (Coelho de Souza and Rodrigues 1969, and references therein). The problem of establishing a hamiltonian formalism has also been discussed within the so called generalized mechanics, which is an analytical mechanics based on lagrangians containing higher derivatives (Borneas 1972). The further question of a consistent quantization procedure has caused several controversies (cf Hayes 1969, Kimura 1972).

The study of generalized lagrangians can also be motivated by the following observation. As is well known, an ordinary differential equation of degree $n > 1$ can be written as a system of coupled first-order equations simply by introducing new functions defined in terms of suitable derivatives of the original function. There are many instances where this process can be reversed, that is, given a system of first-order equations we can eliminate all functions but one to obtain an equation of higher degree in the remaining function. (This is certainly possible for the case of linear first-order differential equation systems with constant coefficients, which will be studied below.) Which representation is preferred varies from problem to problem. However, the important point is, that the corresponding theories of differential equations have both been developed to a high degree of sophistication.

The situation is quite similar within the variational calculus where a lagrangian containing higher derivatives can be reduced to one containing first derivatives only at the expense of an increased number of functions. Conversely, we can use the variational equations to eliminate all functions but one from a lagrangian involving several functions and their first derivatives. (This might not always be possible, cf above.)

In contrast with the theory of ordinary differential equations, all emphasis in the variational calculus has been on the formulation which uses first derivatives only. The purpose of this article is to work towards a more balanced situation in this respect by extending the technique of averaged lagrangians to the case of generalized mechanics. In doing this, it is hoped that the theory of averaged lagrangians will gain more flexibility and admit greater possibilities of choice.

We will show that the conventional expressions for the frequency equation and averaged energy of oscillations in terms of averaged lagrangians are still valid in the generalized case of lagrangians containing higher-order derivatives. This is shown quite generally using the lagrangian multiplier technique. To illustrate a more tangible approach we also give an alternative direct proof for the case of linear lagrangians using a definite representation. Although the lagrangian which corresponds to a certain differential equation is not unique, we prove that the averaged lagrangian is a quantity which is uniquely determined. The relation between the hamiltonian and the energy of the system is discussed. Finally some mechanical applications are given which lead to a formulation in terms of lagrangians with higher-order derivatives and a comparison is made with the standard approach.

2. General theory of averaged lagrangians

Consider a lagrangian of the general form $L = L(x, \mathrm{D}x, \ldots, \mathrm{D}^N x)$ where x is a generalized coordinate function depending upon the variable t and $\mathrm{D}^m x = \mathrm{d}^m x/\mathrm{d}t^m$. The generalized variational equation corresponding to L is (Borneas 1972):

$$\frac{\delta L}{\delta x} = \sum_{n=0}^{N} (-\mathrm{D})^n \left(\frac{\partial L}{\partial(\mathrm{D}^n x)} \right) = 0. \tag{2.1}$$

We shall study lagrangians whose linearized variational equation admits uniform oscillatory solutions, that is, solutions of the form

$$x = \tfrac{1}{2}\{a \exp(\mathrm{i}\omega t) + a^* \exp(-\mathrm{i}\omega t)\} = \mathrm{Re}\,\{a \exp(\mathrm{i}\omega t)\}$$

where the asterisk denotes complex conjugate and a and ω are the constant amplitude and angular frequency respectively of the oscillation. If we insert the uniform solution into the lagrangian and average over one period of oscillation we obtain the averaged lagrangian $\langle L \rangle$ as

$$\langle L \rangle = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} L(x, \mathrm{D}x, \ldots, \mathrm{D}^N x)|_{x=\mathrm{Re}\{a \exp(\mathrm{i}\omega t)\}}\, \mathrm{d}t. \tag{2.2}$$

$\langle L \rangle$ will depend only on the three variables a, a^*, and ω. It is known (see for instance Bretherton and Garrett 1969) that if L contains first derivatives at most of the variable x, then the frequency equation of the oscillation is determined by

$$\frac{\partial\langle L\rangle}{\partial a} = 0 = \frac{\partial\langle L\rangle}{\partial a^*} \tag{2.3}$$

and the averaged hamiltonian $\langle H\rangle$ by

$$\langle H\rangle = \omega\frac{\partial\langle L\rangle}{\partial\omega} - \langle L\rangle. \tag{2.4}$$

We will show that equations (2.3) and (2.4) are still valid when higher-order derivatives are present in L. This can be accomplished by rewriting the general lagrangian in a form for which the original theory is applicable. For that purpose we introduce the new variables $x_n(n = 0, 1, \ldots, N)$ through the relations

$$\begin{cases} \mathrm{D}x_n = x_{n+1} & n = 0, 1, \ldots, N-1 \\ x_0 = x. \end{cases} \tag{2.5}$$

Using the technique of lagrangian multipliers (Gelfand and Fomin 1965) we regard (2.5) as subsidiary conditions and write

$$\begin{aligned} L(x, \mathrm{D}x, \ldots, \mathrm{D}^N x) &= L(x_0, x_1, \ldots, x_N) + \sum_{n=0}^{N-1} \lambda_n(\mathrm{D}x_n - x_{n+1}) \\ &= \hat{L}(x_0, \mathrm{D}x_0, x_1, \mathrm{D}x_1, \ldots, x_N, \lambda_0, \ldots, \lambda_{N-1}) \end{aligned} \tag{2.6}$$

where $\lambda_n(n = 0, 1, \ldots, N-1)$ are the lagrangian multipliers. The lagrangian $\hat{L}$ can be treated by conventional methods and after averaging we obtain

$$\langle L\rangle(a, a^*, \omega) = \langle\hat{L}\rangle(a_n, a_n^*, \Lambda_n, \Lambda_n^*, \omega) \tag{2.7}$$

where

$$x_n = \mathrm{Re}\{a_n \exp(\mathrm{i}\omega t)\} \qquad (n = 0, 1, \ldots, N)$$

and

$$\lambda_n = \mathrm{Re}\{\Lambda_n \exp(\mathrm{i}\omega t)\} \qquad (n = 0, 1, \ldots, N-1).$$

The frequency equation is obtained from the system

$$\begin{aligned} &\frac{\partial\langle\hat{L}\rangle}{\partial a_n} = 0 = \frac{\partial\langle\hat{L}\rangle}{\partial a_n^*} \qquad n = 0, 1, \ldots, N \\ &\frac{\partial\langle\hat{L}\rangle}{\partial \Lambda_n} = 0 = \frac{\partial\langle\hat{L}\rangle}{\partial \Lambda_n^*} \qquad n = 0, 1, \ldots, N-1 \end{aligned} \tag{2.8}$$

and the averaged hamiltonian is given by

$$\langle\hat{H}\rangle = \omega\frac{\partial\langle\hat{L}\rangle}{\partial\omega} - \langle\hat{L}\rangle. \tag{2.9}$$

However, we can also regard equations (2.8) as a system which specifies a_n, a_n^*, Λ_n, and Λ_n^* as functions of $a_0 = a$, $a_0^* = a^*$, and ω, that is,

$$\begin{aligned} &a_n = a_n(a, a^*, \omega) \qquad a_n^* = a_n^*(a, a^*, \omega) \qquad n = 1, 2, \ldots, N \\ &\Lambda_n = \Lambda_n(a, a^*, \omega) \qquad \Lambda_n^* = \Lambda_n^*(a, a^*, \omega) \qquad n = 1, 2, \ldots, N-1. \end{aligned} \tag{2.10}$$

This implies that

$$\frac{\partial\langle L\rangle}{\partial a} = \sum_{n=0}^{N}\left(\frac{\partial\langle\hat{L}\rangle}{\partial a_n}\frac{\partial a_n}{\partial a}+\frac{\partial\langle\hat{L}\rangle}{\partial a_n^*}\frac{\partial a_n^*}{\partial a}\right)+\sum_{n=0}^{N-1}\left(\frac{\partial\langle\hat{L}\rangle}{\partial\Lambda_n}\frac{\partial\Lambda_n}{\partial a}+\frac{\partial\langle\hat{L}\rangle}{\partial\Lambda_n^*}\frac{\partial\Lambda_n^*}{\partial a}\right) = 0 \tag{2.11}$$

by means of equations (2.8). Analogously we obtain

$$\frac{\partial\langle L\rangle}{\partial\omega} = \frac{\partial\langle\hat{L}\rangle}{\partial\omega} \tag{2.12}$$

and consequently

$$\langle H\rangle = \omega\frac{\partial\langle L\rangle}{\partial\omega}-\langle L\rangle = \omega\frac{\partial\langle\hat{L}\rangle}{\partial\omega}-\langle\hat{L}\rangle = \langle\hat{H}\rangle. \tag{2.13}$$

Equations (2.11) and (2.13) show that the conventional expressions for the frequency equation and averaged hamiltonian are still valid for general lagrangians containing higher derivatives.

Conversely, if we start from a lagrangian of the form $\hat{L} = \hat{L}(x_1, \mathrm{D}x_1, x_2, \ldots, x_N, \mathrm{D}x_N)$ and the subsequent averaged form $\langle\hat{L}\rangle(a_n, a_n^*, \omega)$ we can easily obtain an averaged generalized lagrangian $\langle L\rangle$ as follows. The frequency equation system of $\langle\hat{L}\rangle$ is

$$\frac{\partial\langle\hat{L}\rangle}{\partial a_n} = 0 = \frac{\partial\langle\hat{L}\rangle}{\partial a_n^*} \qquad n = 1, 2, \ldots, N. \tag{2.14}$$

We now use equations (2.14) to express all a_n and a_n^* in one preferred coordinate which we call a. This yields

$$a_n = a_n(a, a^*, \omega) \qquad a_n^* = a_n^*(a, a^*, \omega) \qquad n = 1, 2, \ldots, N. \tag{2.15}$$

Inserting these expressions into $\langle\hat{L}\rangle$ we obtain

$$\langle L\rangle(a, a^*, \omega) = \langle\hat{L}\rangle(a_n, a_n^*, \omega). \tag{2.16}$$

Then, in the same way as above, we obtain expressions for the frequency equation and averaged hamiltonian in terms of the generalized lagrangian $\langle L\rangle$. These results are in form identical with equations (2.11) and (2.13). We note, that the procedure leading to equation (2.15) is analogous to that used by Askne (1972b) in his work on quantization of waves in dispersive media in terms of averaged lagrangians.

3. Linear problems in generalized mechanics

In order to make a more detailed study of some aspects of the theory of averaged lagrangians and their connection with generalized mechanics we will restrict the following analysis to linear variational problems, that is, we assume that the variational equation (2.1) is of the form

$$P(\mathrm{D})x = 0 \tag{3.1}$$

where $P(\mathrm{D})$ is a linear self-adjoint differential operator with constant coefficients. Every lagrangian, which corresponds to a linear variational problem as given by equation

(3.1), is of the following general form:

$$L = \sum_{m,n} a_{m,n} \mathrm{D}^m x \mathrm{D}^n x \tag{3.2}$$

where $a_{m,n}$ are constant. We prefer to work with an explicit expression for L although most of the results given below could have been obtained in a form-invariant way.

We have shown (Anderson 1973) that corresponding to every bilinear lagrangian of the form (3.2) there is a lagrangian quadratic in the coordinate function x and its derivatives which is equivalent to the former in the sense that both give rise to the same differential equation. Furthermore the quadratic lagrangian is uniquely determined by the corresponding variational equation. These results are essentially obtained by repeated use of the following identity:

$$\mathrm{D}^m x \mathrm{D}^n x = \mathrm{D}(\mathrm{D}^{m-1} \mathrm{D}^n x) - \mathrm{D}^{m-1} x \mathrm{D}^{n+1} x \tag{3.3}$$

and the well known fact that a total derivative in a lagrangian does not contribute to the variational equation. However, it is trivial to show that $\langle \mathrm{D}^m x \mathrm{D}^n x \rangle = -\langle \mathrm{D}^{m-1} x \mathrm{D}^{n+1} x \rangle$ or equivalently $\langle \mathrm{D}(\mathrm{D}^{m-1} x \mathrm{D}^n x) \rangle = 0$. This implies that all lagrangians, which are equivalent, have the same averaged lagrangian. This result, together with the theory of quantization in terms of averaged lagrangians put forward by Askne (1972b), might have some interesting consequences for the following disputed problem in generalized mechanics. It has turned out that, if L_1 and L_2 are equivalent lagrangians, it might perfectly well happen that the quantization of the corresponding problem in terms of L_1 gives no trouble at all while for L_2 a consistent quantization procedure is very hard to find. This difficulty becomes acute already in the seemingly simple problem of quantization of the harmonic oscillator (see Hayes 1969, Kimura 1972). A quantization theory based on the (unique) averaged lagrangian obviously avoids this problem which is due to the ambiguity of the lagrangian characterizing a certain variational equation.

Resuming our analysis, the averaged lagrangian corresponding to equation (3.1) can be determined as follows. It is easy to show that a possible choice of lagrangian characterizing equation (3.1) is $L = \frac{1}{2} x P(\mathrm{D}) x$ which gives directly

$$\langle L \rangle = \tfrac{1}{4} P(\omega) a a^* \tag{3.4}$$

where $P(\omega)$ denotes $P(\mathrm{D})$ with $\mathrm{D} = \mathrm{i}\omega$. Since $P(\mathrm{D})$ is self-adjoint (see eg Lanczos 1961), we conclude that $P(\omega)$ is real.

We will now proceed to give a direct proof of relations (2.3) and (2.4) for the case of linear, generalized lagrangians. For that purpose we use the equivalent normal form of L mentioned above. This is by no means necessary, but it gives shorter and neater computations than the general bilinear form (equation (3.2)). Thus we have for L

$$L = \tfrac{1}{2} \sum_{n=0}^{N} a_n (\mathrm{D}^n x)^2 \tag{3.5}$$

which yields the averaged lagrangian

$$\langle L \rangle = \tfrac{1}{4} \sum_{n=0}^{N} a_n \omega_n \omega_n^* a a^* \tag{3.6}$$

where we have introduced the notations $\omega_n = (\mathrm{i}\omega)^n$ $(n = 0, 1, \ldots, N)$. This implies that

$$\frac{\partial \langle L \rangle}{\partial a} = 0 = \frac{\partial \langle L \rangle}{\partial a^*} \tag{3.7}$$

is equivalent to $\langle L\rangle = 0$ or inserting the expressions for ω_n we get

$$\sum_{n=0}^{N} a_n \omega^{2n} = 0. \tag{3.8}$$

This constitutes an algebraic equation for ω, that is, the frequency equation.

The hamiltonian H corresponding to a generalized lagrangian is (see Borneas 1972)

$$H = \sum_{m=1}^{N} \mathrm{D}^m x P_m - L \tag{3.9}$$

where the generalized momenta P_m are given by

$$P_m = \sum_{j=0}^{N-m} (-\mathrm{D})^j \left(\frac{\partial L}{\partial(\mathrm{D}^{j+m}x)}\right). \tag{3.10}$$

Some straightforward calculations yield for the average of H

$$\begin{aligned}\langle H\rangle &= \sum_{n=1}^{N} 2n\omega_n \frac{\partial\langle L\rangle}{\partial\omega_n} - \langle L\rangle \\ &= \sum_{n=1}^{N} 2n\omega_n^* \frac{\partial\langle L\rangle}{\partial\omega_n^*} - \langle L\rangle\end{aligned} \tag{3.11}$$

where $\langle L\rangle$ is considered as a function of the variables ω_n, ω_n^*, a, and a^*. This also implies that

$$\begin{aligned}\omega\frac{\partial\langle L\rangle}{\partial\omega} - \langle L\rangle &= \omega \sum_{n=0}^{N}\left(\frac{\partial\langle L\rangle}{\partial\omega_n}\frac{\mathrm{d}\omega_n}{\mathrm{d}\omega} + \frac{\partial\langle L\rangle}{\partial\omega_n^*}\frac{\mathrm{d}\omega_n^*}{\mathrm{d}\omega}\right) - \langle L\rangle \\ &= \sum_{n=1}^{N} n\left(\omega_n\frac{\partial\langle L\rangle}{\partial\omega_n} + \omega_n^*\frac{\partial\langle L\rangle}{\partial\omega_n^*}\right) - \langle L\rangle = \langle H\rangle\end{aligned} \tag{3.12}$$

and we obtain the result (2.13) as we should.

4. Averaged energy density

In this section we will discuss the consequences of the fact that although the hamiltonian H is a constant of the motion, it does not necessarily represent the physical energy of the system. For this to be the case we must have a lagrangian which is energetically correct, an expression which will be clarified below. We introduce for a moment an external force F acting on the system and consider the inhomogeneous equation

$$P(\mathrm{D})x = F. \tag{4.1}$$

x is a displacement coordinate and the power delivered to the system is given by vF, where $v = \mathrm{D}x$ is the velocity (cf Askne 1972a, Bretherton and Garret 1969).

The total lagrangian L of equation (4.1) is the sum of two parts L_h and L_p corresponding to homogeneous and particular parts respectively, that is, L_h is a bilinear lagrangian of the form studied in the preceding paragraphs and $L_\mathrm{p} = +xF$. Averaging L we obtain

$$\langle L\rangle = \langle L_\mathrm{h}\rangle + \langle xF\rangle. \tag{4.2}$$

This situation is implicity assumed when one concludes that the averaged energy $\langle W\rangle$

of the free oscillations ($F = 0$) is given by the averaged hamiltonian $\langle H \rangle$. A direct proof of this is given in appendix 1. Then we have

$$\langle W \rangle = \langle H \rangle = \omega \frac{\partial \langle L_{\mathrm{h}} \rangle}{\partial \omega} \tag{4.3}$$

where we have used the fact that $\langle L_{\mathrm{h}} \rangle = 0$ for free oscillations. The distinction between $\langle W \rangle$ and $\langle H \rangle$ is decisive, for example, in the following situation. Consider a general case of coupled equations as given by the matrix equation

$$\mathbf{P}(\mathrm{D})\boldsymbol{x} = \boldsymbol{F} \tag{4.4}$$

where $\mathbf{P}(\mathrm{D})$ is a linear matrix operator and $\boldsymbol{x}$ and $\boldsymbol{F}$ are vectors specifying the displacement coordinates and the external forces respectively. If we eliminate all equations but one and keep only the force F corresponding to the remaining variable x, we obtain

$$Q(\mathrm{D})x = R(\mathrm{D})F \tag{4.5}$$

because generally we must operate on F with some differential operator $R(\mathrm{D})$ in order to obtain the desired elimination. As before equation (4.5) is assumed self-adjoint, which implies that $R(\mathrm{D})$ is a self-adjoint operator. However, in this case we have for the average of L_{p}

$$\langle L_{\mathrm{p}} \rangle = \langle xR(\mathrm{D})F \rangle = R(\omega)\langle xF \rangle \tag{4.6}$$

which implies that

$$\langle W \rangle = \frac{1}{R(\omega)}\langle H \rangle = \frac{\omega}{R(\omega)} \frac{\partial \langle L_{\mathrm{h}} \rangle}{\partial \omega} = \omega \frac{\partial \langle L'_{\mathrm{h}} \rangle}{\partial \omega} \tag{4.7}$$

where we have introduced the notation $\langle L'_{\mathrm{h}} \rangle = \langle L_{\mathrm{h}} \rangle / R(\omega)$. Note that $\langle L_{\mathrm{h}} \rangle$ and $\langle L'_{\mathrm{h}} \rangle$ both yield the same frequency equation but not the same averaged hamiltonian. This gives by means of equation (2.10)

$$\langle W \rangle = -\frac{\omega}{4} \frac{\partial}{\partial \omega} \left(\frac{Q(\omega)}{R(\omega)} \right) aa^*. \tag{4.8}$$

Finally we point out that the analysis is not restricted to a mechanical context, but should be applicable to all kinds of wave motion where it is possible to find a pair (x, F) such that x characterizes the wave motion and F the external source and the product $\mathrm{D}x$. F gives the power delivered to the system (cf Askne 1972a).

5. Applications

As a simple application we shall study the following example, which has the advantage of making possible a direct comparison with the standard procedure.

Consider the harmonic oscillator

$$-m\mathrm{D}^2 x - kx = 0. \tag{5.1}$$

The lagrangian of this problem is conventionally taken to be

$$L_1 = \tfrac{1}{2}m(\mathrm{D}x)^2 - \tfrac{1}{2}kx^2 \tag{5.2}$$

which gives the averaged lagrangian

$$\langle L_1 \rangle = \frac{m}{4}\left(\omega^2 - \frac{k}{m}\right) aa^*. \tag{5.3}$$

The frequency equation is

$$\frac{\partial \langle L_1 \rangle}{\partial a} = 0 \Leftrightarrow \omega^2 = \frac{k}{m} = \omega_0^2. \tag{5.4}$$

The averaged stored energy is obtained as

$$\langle W_1 \rangle = \langle H_1 \rangle = \left(\omega \frac{\partial \langle L_1 \rangle}{\partial \omega}\right)_{\omega = \omega_0} = \frac{m\omega_0^2}{2} aa^*. \tag{5.5}$$

However, a lagrangian characterizing equation (5.1) could equally well be taken as (Hayes 1969):

$$L = -\tfrac{1}{2}mx\mathrm{D}^2x - \tfrac{1}{2}kx^2 \tag{5.6}$$

which contains the second derivative of x. (Quantization in terms of this lagrangian has proved very intricate.) By means of the analysis presented here we see that $\langle L_1 \rangle = \langle L \rangle$ since L and L_1 only differ by a total derivative ($L_1 = L + \tfrac{1}{2}m\mathrm{D}(x\mathrm{D}x)$) and the results obtained from $\langle L \rangle$ are consequently the same as those above.

A less trivial example is the following. Consider a general undamped system with two degrees of freedom, for example, consisting of two masses m_1 and m_2 suspended by springs (k_1 and k_2) and tied together by a coupling spring (k_3). x_1 and x_2 denote the displacement coordinates. The equations of motion are (see eg den Hartog 1947):

$$\begin{aligned} -m_1\mathrm{D}^2x_1 - (k_1+k_3)x_1 + k_3x_2 &= 0 \\ -m_2\mathrm{D}^2x_2 - (k_2+k_3)x_2 + k_3x_1 &= 0. \end{aligned} \tag{5.7}$$

The lagrangian of the equation system (5.7) is

$$\begin{aligned} \hat{L} &= \hat{L}(x_1, \mathrm{D}x_1, x_2, \mathrm{D}x_2) \\ &= \tfrac{1}{2}\{m_1(\mathrm{D}x_1)^2 + m_2(\mathrm{D}x_2)^2 - (k_1+k_3)x_1^2 - (k_2+k_3)x_2^2 + 2k_3x_1x_2\} \end{aligned} \tag{5.8}$$

with the corresponding averaged lagrangian

$$\langle \hat{L} \rangle = \tfrac{1}{4}\{(m_1\omega^2 - k_1 - k_3)a_1a_1^* + (m_2\omega^2 - k_2 - k_3)a_2a_2^* + k_3(a_1a_2^* + a_1^*a_2)\}. \tag{5.9}$$

The frequency equation system is

$$\begin{aligned} \frac{\partial \langle \hat{L} \rangle}{\partial a_1^*} &= 0 \Rightarrow (m_1\omega^2 - k_1 - k_3)a_1 + k_3a_2 = 0 \\ \frac{\partial \langle \hat{L} \rangle}{\partial a_2^*} &= 0 \Rightarrow (m_2\omega^2 - k_2 - k_3)a_2 + k_3a_1 = 0. \end{aligned} \tag{5.10}$$

The vanishing of the secular determinant yields the frequency equation

$$M^2(\omega^4 - \kappa\omega^2 + \alpha) = 0 \tag{5.11}$$

where $M^2 = m_1m_2$, $\kappa = (k_1+k_3)/m_1 + (k_2+k_3)/m_2$, and $\alpha^2 = (k_1k_2 + k_1k_3)/M^2$. The averaged energy is

$$\langle W \rangle = \omega \frac{\partial \langle \hat{L} \rangle}{\partial \omega} - \langle \hat{L} \rangle = \tfrac{1}{2}\omega^2(m_1 a_1 a_1^* + m_2 a_2 a_2^*) \tag{5.12}$$

or in terms of coordinate a_1

$$\langle W \rangle = \frac{m_1 \kappa}{2} \frac{\omega^2 - 2\alpha^2/\kappa}{\omega^2 - (k_2 + k_3)/m_2} a_1 a_1^*. \tag{5.13}$$

On the other hand, if we eliminate the coordinate x_2 from the system (5.7) we obtain

$$Q(\mathrm{D})x_1 = R(\mathrm{D})F_1 \tag{5.14}$$

where we have introduced an external force F_1 in accordance with the normalization procedure discussed in the previous paragraph. $Q(\mathrm{D})$ and $R(\mathrm{D})$ are given by

$$Q(\mathrm{D}) = -M^2(\mathrm{D}^4 + \kappa \mathrm{D}^2 + \alpha^2); \qquad R(\mathrm{D}) = -\{m_2 \mathrm{D}^2 + (k_2 + k_3)\}. \tag{5.15}$$

A lagrangian corresponding to the homogeneous part is given by

$$L_{\mathrm{h}}(x_1, \mathrm{D}x_1, \mathrm{D}^2 x_1) = \tfrac{1}{2}M^2\{(\mathrm{D}^2 x_1)^2 - \kappa(\mathrm{D}x_1)^2 + \alpha^2 x_1^2\} \tag{5.16}$$

which contains the second derivative of x_1. The average of L_{h} is

$$\langle L_{\mathrm{h}} \rangle = \tfrac{1}{4}M^2(\omega^4 - \kappa\omega^2 + \alpha^2)a_1 a_1^* \tag{5.17}$$

and using the notations of § 4 we obtain

$$\langle L_{\mathrm{h}}' \rangle = \frac{M^2(\omega^4 - \kappa\omega^2 + \alpha^2)}{4\{m_2\omega^2 - (k_2 + k_3)\}} a_1 a_1^*. \tag{5.18}$$

It is now easy to show that the frequency equation and averaged energy obtained from $\langle L_{\mathrm{h}}' \rangle$ coincide with the result given in equations (5.11) and (5.13).

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Appendix 1

We will show here that the hamiltonian H corresponding to equation (4.1) is directly related to the stored energy W of the system. If we multiply equation (4.1) with $\mathrm{D}x$ we obtain

$$\mathrm{D}xP(\mathrm{D})x = F\mathrm{D}x. \tag{A.1}$$

The right-hand side of equation (A.1) is the power delivered to the system and if we can write the left-hand side as $\mathrm{D}W$, then equation (A.1) becomes a conservation equation, which makes it possible to identify W with the stored energy of the system. We can rewrite the left-hand side of equation (A.1) as follows:

$$\mathrm{D}xP(\mathrm{D})x = \mathrm{D}(xP(\mathrm{D})x) - x\mathrm{D}P(\mathrm{D})x. \tag{A.2}$$

By means of the Lagrange identity (see eg Ince 1944), we get

$$xDP(D)x = DC(x, x) + x\overline{DP}(D)x \qquad (A.3)$$

where $C(x, x)$ is the bilinear concomitant corresponding to the operator $DP(D)$ and $\overline{DP}(D)$ the adjoint operator. However, since $P(D)$ is self-adjoint we get

$$\overline{DP}(D) = -DP(D)$$

which implies

$$xDP(D)x = \tfrac{1}{2}DC(x, x). \qquad (A.4)$$

Using this result together with equation (A.2) we obtain

$$W = xP(D)x - \tfrac{1}{2}C(x, x). \qquad (A.5)$$

This expression will be put into a more familiar form. Write $P(D)$ as

$$P(D) = \sum_{n=0}^{N} b_n D^n. \qquad (A.6)$$

Then the concomitant is given by (cf Ince 1944)

$$C(x, x) = \sum_{n=1}^{N} b_n \sum_{k=0}^{n} ((-D)^k x)(D^{n-k} x). \qquad (A.7)$$

Equation (A.5) then becomes

$$W = \tfrac{1}{2}\left(xP(D)x - \sum_{n=1}^{N} b_n \sum_{k=1}^{n} ((-D)^k x)(D^{n-k} x)\right). \qquad (A.8)$$

But from appendix 1 and equation (4.2) we get $L_h = -\tfrac{1}{2}xP(D)x$ and after doing some algebra including a change of summation variables we obtain

$$W = -L_h + \sum_{n=1}^{N} D^n x \sum_{k=1}^{n} (-1)^{n+k}(-D)^k \frac{\partial L_h}{\partial(D^{n+k})}. \qquad (A.9)$$

But since $P(D)$ is self-adjoint, $b_n = 0$ for odd n, which implies

$$\frac{\partial L_h}{\partial(D^{n+k}x)} = 0 \qquad \text{for } n+k \text{ odd} \qquad (A.10)$$

and we obtain (see equations (3.7) and (3.8))

$$W = \sum_{n=1}^{N} D^n x P_n - L_h = H \qquad (A.11)$$

where P_n are the generalized momenta corresponding to L_h.

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